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Journal of Computational Physics 187 (2003) 428–440
JOURNAL OF COMPUTATIONAL PHYSICS
www.elsevier.com/locate/jcp

Positive schemes for solving multi-dimensional hyperbolic systems of conservation laws II

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Received 7 November 2001; received in revised form 28 January 2003; accepted 13 February 2003

Dedicated to Robert Richtmyer at his 90th birthday with affection and admiration

Abstract

Kurganov and Tadmor have developed a numerical scheme for solving the initial value problem for hyperbolic systems of conservation laws. They showed that in the scalar case their scheme satisfies a local maximum–minimum principle i.e., the solution at future is bounded above and below by the solution at current locally. In this paper we show that this scheme is positive in the sense of Friedrichs for systems as well. We present the scheme of Kurganov and Tadmor as a convex combination of composites of positive schemes. Since each component of a composite scheme is bounded in the l^2 norm, so is the convex combination of the composites. To achieve second order accuracy in time, we use a Runge–Kutta type scheme due to Shu and Osher. We present two numerical experiments to add to the ones carried out by Kurganov and Tadmor.

1. Introduction

The total variation of a solution of a single conservation law in one space variable is a diminishing function of time. For hyperbolic systems of conservation laws this is no longer strictly true, although the related Glimm functional is decreasing. Harten, in [3], and many after him, have derived total variation diminishing (TVD) schemes that worked splendidly for approximating solutions of systems of conservation laws, principally the Euler equations of compressible flow in one space variable.

The total variation of solutions of hyperbolic systems of conservation laws in more than one space variable is not a diminishing function of times; in fact it can become unbounded because of the possibility of focusing. Therefore, there can be no TVD schemes for such problems. The only functional known to be

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[1] The author was supported by National Science Foundation Grants DMS-9112654 and DMS-9406467, and Air Force Grant AFOSR F49620-94-1-0132.

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doi:10.1016/S0021-9991(03)00100-1

bounded for linear systems of equations that are symmetric hyperbolic is energy, represented by the L^2 norm of the solutions. Friedrichs has shown that difference approximations to solutions of such systems which employ coefficient matrices that are symmetric, positive definite, and depend Lipschitz continuously on the space variables, are bounded under the discrete l^2 norm of the numerical solutions. This suggests positivity as a design principle for solving systems of conservation laws in more than one space variable.

The organization of this paper is as follows. In the beginning of Section 2, we review the positivity principle for linear systems. In Section 2.1 we describe the notion of a positive scheme for solving symmetric nonlinear systems. In Section 2.2 we describe the scheme of Kurganov and Tadmor for one space dimension and prove that the scheme is a convex combination of composites of positive schemes. The Runge–Kutta method to achieve second order accuracy in time is described in Section 2.3. In Section 2.4 we explain how to combine the fluxes in all space direction. The numerical experiments are described in Section 3.

2. Positive scheme

We consider multi-dimensional hyperbolic systems of conservation laws

$$U_t + \sum_{s=1}^{d} F_s(U)_{x_s} = 0, \tag{1}$$

where $U = (u_1, \dots, u_n)^{\mathrm{T}} \in \mathbf{R^n}$ and $x = (x_1, x_2, \dots, x_d) \in \mathbf{R^d}$. We assume that all Jacobian matrices $A_s = \partial F_s(U)/\partial U$ are symmetric or simultaneously symmetrizable with the same similarity transformation for the proof of positivity [11]. The actual scheme does not required that [9].

A linear symmetric hyperbolic systems is of the form

$$U_t + \sum_{s=1}^{d} A_s U_{x_s} = 0, \tag{2}$$

A_s real symmetric matrices, which is independent from U and its dependence on x is Lipschitz continuous. It is easy to show in this case that the L^2 norm of a solution is of bounded growth

$$\|U(t)\| \leqslant e^{ct}\|U(0)\|, \tag{3}$$

where $\|V(t)\|^2 = \int (V(x,t), V(x,t))\,\mathrm{d}x$ and c is related to the Lipschitz constant.

Set a uniform Cartesian grid $\{\Omega_J\}$ in $\mathbf{R^d}$, where $J = (j_1, j_2, \dots, j_d)$ is a lattice point in which all j_s are integers. The grid points are $x_J = (j_1\Delta x_1, \dots, j_d\Delta x_d)$. Let U_J be an approximation to the value of the solution $U(x_J, t)$ at current time t, and U_J^* an approximation to the value of the future solution $U(x_J, t + \Delta t)$.

Friedrichs [2] has shown that solutions of such Eq. (2) can be approximated by solutions of difference equations of form

$$U_J^* = \sum_K C_{J,K} U_{J+K}, \tag{4}$$

and with the coefficient matrices $C_{J,K}$ satisfying the following properties:

$$\begin{aligned}
&\text{(i)}\ C_{J,K} \text{ is symmetric and semi-positive definite,}\\
&\text{(ii)}\ \sum_K C_{J,K} = I, \quad \text{where } I \text{ is the identity matrix,}\\
&\text{(iii)}\ C_{J,K} = 0 \text{ except for a finite set of } K,\\
&\text{(iv)}\ C_{J,K} \text{ depends Lipschitz continuously on } x.
\end{aligned} \tag{5}$$

The first two properties, in scalar cases, imply that the solution at future U^* is a convex combination of the solution at current U, which leads to a local maximum–minimum principle. The third property echos the fact that the propagation speed of waves is finite for hyperbolic systems.

He has shown in [2] that the l^2 norm of solutions of difference schemes satisfying these properties has bounded growth:

$$\|U^*\| \leqslant (1 + \text{const}\varDelta)\|U\|, \tag{6}$$

where the discrete l^2 norm is defined as

$$\|U\|^2 = \sum_J (U_J, U_J).$$

The value of the constant in (6) depends on the Lipschitz constant.

Proof of (6): Take the scalar product of (4) with U_J^*

$$(U_J^*, U_J^*) = \sum_K (U_J^*, C_{J,K} U_{K+J}). \tag{7}$$

Since $C_{J,K}$ is symmetric and semi-positive definite, $(U, C_{J,K} V)$ can be regarded as an inner product, to which the Schwartz inequality applies; combined with the inequality between arithmetic and geometric mean we get

$$(U, C_{J,K} V) \leqslant \sqrt{(U, C_{J,K} U)}\sqrt{(V, C_{J,K} V)} \leqslant \frac{1}{2}(U, C_{J,K} U) + \frac{1}{2}(V, C_{J,K} V). \tag{8}$$

Using this on the right side of (7) gives

$$(U_J^*, U_J^*) \leqslant \frac{1}{2}\sum_K (U_J^*, C_{J,K} U_J^*) + \frac{1}{2}\sum_K (U_{J+K}, C_{J,K} U_{J+K}). \tag{9}$$

Carrying out the summation with respect to K, using condition (ii) of (5) and multiplying by 2 gives

$$(U_J^*, U_J^*) \leqslant \sum_K (U_{J+K}, C_{J,K} U_{J+K}). \tag{10}$$

Now sum with respect to J, and introduce $K + J = N$ as new index of summation:

$$\sum_J (U_J^*, U_J^*) \leqslant \sum_{N,K} (U_N, C_{N-K,K} U_N). \tag{11}$$

Because Lipschitz continuity of the coefficient matrices and the fact that K ranges over a finite stencil,

$$C_{N-K,K} = C_{N,K} + D_{N,K},$$

and

$$\rho(D_{N,K}) = \mathrm{O}(\varDelta),$$

where $\rho(D)$ is the spectral radius of D and $\Delta = \min(\Delta t, \Delta x_1, \ldots, \Delta x_d)$. Therefore, we get that the right side of (11) is

$$\leqslant \sum_{N,K} (U_N, C_{N,K} U_N) + \mathrm{O}(\varDelta) \sum_N (U_N, U_N),$$

which, using (ii) of (5), is equal to

$$\sum_N (U_N, U_N)(1 + \mathrm{O}(\Delta)).$$

Setting this into (11) gives (6)

$$\|U^*\| \leqslant (1 + \mathrm{const}\Delta)\|U\|.$$

Q.e.d.

In the usual variables the Euler equations are not symmetric but symmetrizable [5] and [20]. It is worth remarking that if a system of conservation laws has a convex entropy, then the linearized system is symmetrizable. We have indicated in [11] how to extend the notion of positive schemes to symmetrizable systems, and show their l^2 boundedness.

The difference schemes studied in this paper are nonlinear; when we write them in the form (4), the coefficient matrices $C_{J,K}$ depend on the solution being computed. In the applications of interest these solutions contain shocks and contact discontinuities; therefore, the coefficient matrices $C_{J,K}$ not only fail to be Lipschitz continuous, they are not even continuous. So the analysis of the boundedness of the l^2 norm of the solution is not applicable. A possible way of salvaging our argument is to note that in inequality (8) the left side is substantially smaller than the right side, unless the vector U and V are nearly equal. For unless the vectors U and V are nearly proportional, the Schwartz inequality is a strict inequality. Similarly, unless (U, CU) and (V, CV) are nearly equal, their geometric mean is substantially less than their arithmetic mean. This shows that at a discontinuity the left side of (10) is substantially less than the right side. We do not at this moment see how to show that this gain is enough to counterbalance what we may lose when we replace on the right in inequality (11) the matrix $C_{N-K,K}$ by $C_{N,K}$, but at least we have found a plausible reason why our scheme is as stable as it appears to be in numerical experiments.

2.1. Positivity principle

Conservative schemes are of the form

$$U_J^* = U_J - \sum_{s=1}^{d} \frac{\Delta t}{\Delta x_s}\left[F_{J+\frac{1}{2}e_s} - F_{J-\frac{1}{2}e_s}\right], \tag{12}$$

where Δt is the time step, Δx_s is the spatial step in the x_s dimension and e_s is the unit vector in the x_s direction. A conservative scheme (12) is called *positive* if it could be rewritten in form of (4) and its coefficient matrices satisfy the first three conditions of (5), see [11]. The positivity principle presented above is for the stability of numerical solutions of general multi-dimensional hyperbolic systems. It does not guarantee the positivity of pressure and density for the Euler equation or MHD.

2.2. Positive scheme in one-dimension

Consider one-dimensional hyperbolic systems of conservation laws

$$U_t + F(U)_x = 0. \tag{13}$$

We interpret the scheme of Kurganov and Tadmor as flux splittings, see [16] by Shu and Osher, and Convex ENO scheme [12] by Liu and Osher, and rewrite the Eq. (13) as

$$U_t + F^+(U)_x + F^-(U)_x = 0. \tag{14}$$

Here

$$F(U) = F^{+}(U) + F^{-}(U), \tag{15}$$

and

$$A^{+} = \frac{\partial F^{+}(U)}{\partial U} \geqslant 0 \quad \text{and} \quad A^{-} = \frac{\partial F^{-}(U)}{\partial U} \leqslant 0, \tag{16}$$

where $A^{+} = (\partial F^{+}(U)/\partial U) \geqslant 0$ means symmetric semi-positive definite and $A^{-} = (\partial F^{-}(U)/\partial U \leqslant 0)$ means symmetric semi-negative definite. A simple example is

$$F^{+}(U) = \frac{F(U) + \alpha U}{2} \quad \text{and} \quad F^{-}(U) = \frac{F(U) - \alpha U}{2}, \tag{17}$$

where α is a function of x and t (but not U) and $\alpha \geqslant \rho(\partial F/\partial U)$, where $\rho(\partial F/\partial U)$ is the spectral radius of the local Jacobian matrix of $F(U)$. This is called local flux splitting. If α is taken to be constant over the whole domain in x, the splitting is called global flux splitting. The local one produces less numerical viscosity than the global one. The importance and potential of flux splitting started to be realized and established in the early through mid eighties, [1,4,16,18,21,22], etc. and an early work can be traced back to Steger in late seventies [17].

We consider schemes in conservation form, i.e.,

$$U_j^{*} = U_j - \lambda[F^{+}_{j+\frac{1}{2}} - F^{+}_{j-\frac{1}{2}} + F^{-}_{j+\frac{1}{2}} - F^{-}_{j-\frac{1}{2}}] \quad \text{where} \quad \lambda = \frac{\Delta t}{\Delta x}. \tag{18}$$

Choice of $F^{+}_{j+\frac{1}{2}}$ is a variant of the ones used by Liu and Lax in [11], and Liu and Osher in [12]. It makes use of two limiting matrices $\Phi^{+}_{j+\frac{1}{2}}$ and $\Psi^{+}_{j+\frac{1}{2}}$ defined as follows:

$$\Phi^{+}_{j+\frac{1}{2}} = \begin{pmatrix} \phi(\theta_1^{+}) & & \\ & \ddots & \\ & & \phi(\theta_n^{+}) \end{pmatrix} \quad \text{and} \quad \Psi^{+}_{j+\frac{1}{2}} = \begin{pmatrix} \phi(\theta_1^{+})/\theta_1^{+} & & \\ & \ddots & \\ & & \phi(\theta_n^{+})/\theta_n^{+} \end{pmatrix}, \tag{19}$$

where the limiter function ϕ, see [19], satisfies

$$0 \leqslant \phi(\theta) \leqslant 2, \quad 0 \leqslant \frac{\phi(\theta)}{\theta} \leqslant 2, \quad \text{and} \quad \phi(1) = 1, \tag{20}$$

and its arguments are

$$\theta_i^{+} = \frac{(U_{j+1} - U_j)_i}{(U_j - U_{j-1})_i}, \quad 1 \leqslant i \leqslant n. \tag{21}$$

Here $(V)_i$ means the ith component of vector V. It follows from (20) that:

$$0 \leqslant \Phi^{+}_{j+\frac{1}{2}} \leqslant 2I \quad \text{and} \quad 0 \leqslant \Psi^{+}_{j+\frac{1}{2}} \leqslant 2I \tag{22}$$

and

$$\Phi^{+}_{j+\frac{1}{2}}(U_j - U_{j-1}) = \Psi^{+}_{j+\frac{1}{2}}(U_{j+1} - U_j). \tag{23}$$

Define

$$U^+_{j+\frac{1}{2}} = U_j + \frac{1}{2}\Phi^+_{j+\frac{1}{2}}(U_j - U_{j-1}) \tag{24}$$

$$F^+_{j+\frac{1}{2}} = F^+\left(U^+_{j+\frac{1}{2}}\right). \tag{25}$$

Because of the relation (23) of $\Phi^+_{j+\frac{1}{2}}$ and $\Psi^+_{j+\frac{1}{2}}$, Eq. (24) can be rewritten as

$$U^+_{j+\frac{1}{2}} = U_j + \frac{1}{2}\Psi^+_{j+\frac{1}{2}}(U_{j+1} - U_j). \tag{26}$$

In smooth regions except at extrema of solutions, all $\theta^+_i = 1 + O(\Delta x)$, hence $\Psi^+_{j+\frac{1}{2}} = I + O(\Delta x)$. Therefore, it follows from (24) that:

$$U^+_{j+\frac{1}{2}} = U_j + \frac{U_j - U_{j-1}}{\Delta x}\frac{\Delta x}{2} + O(\Delta x^2) = U_j + U_x(x_j, t)\frac{\Delta x}{2} + O(\Delta x^2),$$

which is a second order accurate approximation of $U(x_j + \frac{\Delta x}{2}, t)$, and hence $F^+_{j+\frac{1}{2}}$ is a second order accurate approximation of $F^+(U(x_j + \frac{\Delta x}{2}, t))$.

Similarly define the limiting matrices $\Phi^-_{j-\frac{1}{2}}$ and $\Psi^-_{j-\frac{1}{2}}$ as follows:

$$\Phi^-_{j-\frac{1}{2}} = \begin{pmatrix} \phi(\theta^-_1) & & \\ & \ddots & \\ & & \phi(\theta^-_n) \end{pmatrix} \quad \text{and} \quad \Psi^-_{j-\frac{1}{2}} = \begin{pmatrix} \phi(\theta^-_1)/\theta^-_1 & & \\ & \ddots & \\ & & \phi(\theta^-_n)/\theta^-_n \end{pmatrix} \tag{27}$$

and

$$\theta^-_i = (U_j - U_{j-1})_i/(U_{j+1} - U_j)_i, \quad 1 \leqslant i \leqslant n. \tag{28}$$

It follows from (20) that

$$0 \leqslant \Phi^-_{j-\frac{1}{2}} \leqslant 2I \quad \text{and} \quad 0 \leqslant \Psi^-_{j-\frac{1}{2}} \leqslant 2I \tag{29}$$

and

$$\Phi^-_{j-\frac{1}{2}}(U_{j+1} - U_j) = \Psi^-_{j-\frac{1}{2}}(U_j - U_{j-1}). \tag{30}$$

Define

$$U^-_{j-\frac{1}{2}} = U_j - \frac{1}{2}\Phi^-_{j-\frac{1}{2}}(U_{j+1} - U_j) \tag{31}$$

$$F^-_{j-\frac{1}{2}} = F^-\left(U^-_{j-\frac{1}{2}}\right). \tag{32}$$

Because of relation (30), Eq. (31) can be rewritten as

$$U^-_{j-\frac{1}{2}} = U_j - \frac{1}{2}\Psi^-_{j-\frac{1}{2}}(U_j - U_{j-1}). \tag{33}$$

By (26) and (33)

$$U^+_{j+\frac{1}{2}} = \left(I - \frac{1}{2}\Psi^+_{j+\frac{1}{2}}\right)U_j + \frac{1}{2}\Psi^+_{j+\frac{1}{2}}U_{j+1} \quad \text{and} \quad U^-_{j-\frac{1}{2}} = \left(I - \frac{1}{2}\Psi^-_{j-\frac{1}{2}}\right)U_j + \frac{1}{2}\Psi^-_{j-\frac{1}{2}}U_{j-1}. \tag{34}$$

Assume that the system (13) is symmetric. In the following, we prove that the scheme (18,24,25,31,32) is positive.

Because of (22) and (29), the above Eq. (34) shows that $U^+ = \{U^+_{j+\frac{1}{2}}\}$ and $U^- = \{U^-_{j-\frac{1}{2}}\}$ are positive combinations of $U = \{U_j\}$ of form (4) where the conditions (i), (ii), and (iii) of (5) are satisfied. If condition (iv) were satisfied, we could conclude that

$$\|U^+\| \leqslant (1 + \mathrm{O}(\Delta))\|U\| \quad \text{and} \quad \|U^-\| \leqslant (1 + \mathrm{O}(\Delta))\|U\|. \tag{35}$$

Setting (34) into (18) we get

$$U^*_j = U_j - \lambda\left[F^+\left(U^+_{j+\frac{1}{2}}\right) - F^+\left(U^+_{j-\frac{1}{2}}\right) + F^-\left(U^-_{j+\frac{1}{2}}\right) - F^-\left(U^-_{j-\frac{1}{2}}\right)\right]. \tag{36}$$

Recall (17) that

$$F^+\left(U^+_{j+\frac{1}{2}}\right) = \frac{F\left(U^+_{j+\frac{1}{2}}\right) + \alpha_{j+\frac{1}{2}}U^+_{j+\frac{1}{2}}}{2}, \quad F^+\left(U^+_{j-\frac{1}{2}}\right) = \frac{F\left(U^+_{j-\frac{1}{2}}\right) + \alpha_{j-\frac{1}{2}}U^+_{j-\frac{1}{2}}}{2},$$

$$F^-\left(U^-_{j+\frac{1}{2}}\right) = \frac{F\left(U^-_{j+\frac{1}{2}}\right) - \alpha_{j+\frac{1}{2}}U^-_{j+\frac{1}{2}}}{2}, \quad F^-\left(U^-_{j-\frac{1}{2}}\right) = \frac{F\left(U^-_{j-\frac{1}{2}}\right) - \alpha_{j-\frac{1}{2}}U^-_{j-\frac{1}{2}}}{2}.$$

Here α should be chosen to be large enough and will be specified below (41). (36) can be rewritten as

$$U^*_j = U_j - \lambda\left[\frac{F\left(U^+_{j+\frac{1}{2}}\right) + \alpha_{j+\frac{1}{2}}U^+_{j+\frac{1}{2}}}{2} - \frac{F\left(U^+_{j-\frac{1}{2}}\right) + \alpha_{j-\frac{1}{2}}U^+_{j-\frac{1}{2}}}{2} + \frac{F\left(U^-_{j+\frac{1}{2}}\right) - \alpha_{j+\frac{1}{2}}U^-_{j+\frac{1}{2}}}{2} - \frac{F\left(U^-_{j-\frac{1}{2}}\right) - \alpha_{j-\frac{1}{2}}U^-_{j-\frac{1}{2}}}{2}\right]. \tag{37}$$

Let A_j and $\mathscr{A}_j$ be the Roe matrices of F i.e.,

$$F\left(U^+_{j+\frac{1}{2}}\right) - F\left(U^+_{j-\frac{1}{2}}\right) = A_j\left(U^+_{j+\frac{1}{2}} - U^+_{j-\frac{1}{2}}\right),$$

$$F\left(U^-_{j+\frac{1}{2}}\right) - F\left(U^-_{j-\frac{1}{2}}\right) = \mathscr{A}_j\left(U^-_{j+\frac{1}{2}} - U^-_{j-\frac{1}{2}}\right).$$

We then get

$$\begin{aligned} U^*_j = {} & \frac{1}{2}\left\{U_j - \lambda\left[A_j\left(U^+_{j+\frac{1}{2}} - U^+_{j-\frac{1}{2}}\right) + \alpha_{j+\frac{1}{2}}U^+_{j+\frac{1}{2}} - \alpha_{j-\frac{1}{2}}U^+_{j-\frac{1}{2}}\right]\right\} + \frac{1}{2}\left\{U_j - \lambda\left[\mathscr{A}_j\left(U^-_{j+\frac{1}{2}} - U^-_{j-\frac{1}{2}}\right)\right.\right. \\ & \left.\left. - \alpha_{j+\frac{1}{2}}U^-_{j+\frac{1}{2}} + \alpha_{j-\frac{1}{2}}U^-_{j-\frac{1}{2}}\right]\right\}. \end{aligned} \tag{38}$$

Here we have used Roe's mean value theorem [15] which asserts that for any pair of vectors V and W, $F(V) - F(W) = A(M)(V - W)$, where $A = (\partial F(U)/\partial U)$, and $M = M(V, W)$. We emphasize that we do not need to calculate the Roe matrix $A(M)$, we only need it for the proof of positivity.

Using (24) and (31) we can rewrite (38) as

$$\begin{aligned} U_j^* &= \frac{1}{2}\left\{\frac{1}{2}U^+_{j+\frac{1}{2}} - \lambda\left[A_j\left(U^+_{j+\frac{1}{2}} - U^+_{j-\frac{1}{2}}\right) + \alpha_{j+\frac{1}{2}}U^+_{j+\frac{1}{2}} - \alpha_{j-\frac{1}{2}}U^+_{j-\frac{1}{2}}\right] + \frac{1}{2}U_j - \frac{1}{4}\Phi^+_{j+\frac{1}{2}}(U_j - U_{j-1})\right\} \\ &\quad + \frac{1}{2}\left\{\frac{1}{2}U^-_{j-\frac{1}{2}} - \lambda\left[\mathscr{A}_j\left(U^-_{j+\frac{1}{2}} - U^-_{j-\frac{1}{2}}\right) - \alpha_{j+\frac{1}{2}}U^-_{j+\frac{1}{2}} + \alpha_{j-\frac{1}{2}}U^-_{j-\frac{1}{2}}\right] + \frac{1}{2}U_j + \frac{1}{4}\Phi^-_{j-\frac{1}{2}}(U_{j+1} - U_j)\right\} \\ &= \frac{1}{4}\left\{(I - 2\lambda(A_j + \alpha_{j+\frac{1}{2}}I))U^+_{j+\frac{1}{2}} + 2\lambda(A_j + \alpha_{j-\frac{1}{2}}I)U^+_{j-\frac{1}{2}}\right\} \\ &\quad + \frac{1}{4}\left\{(I + 2\lambda(\mathscr{A}_j - \alpha_{j-\frac{1}{2}}I))U^-_{j-\frac{1}{2}} + 2\lambda(-\mathscr{A}_j + \alpha_{j+\frac{1}{2}}I)U^-_{j+\frac{1}{2}}\right\} \\ &\quad + \frac{1}{2}\left\{\left(I - \frac{1}{4}\Phi^+_{j+\frac{1}{2}} - \frac{1}{4}\Phi^-_{j-\frac{1}{2}}\right)U_j + \frac{1}{4}\Phi^+_{j+\frac{1}{2}}U_{j-1} + \frac{1}{4}\Phi^-_{j-\frac{1}{2}}U_{j+1}\right\} = \frac{1}{4}V_j^+ + \frac{1}{4}V_j^- + \frac{1}{2}V_j. \end{aligned} \tag{39}$$

Here

$$\begin{aligned} V_j^+ &= \left(I - 2\lambda(A_j + \alpha_{j+\frac{1}{2}}I)\right)U^+_{j+\frac{1}{2}} + 2\lambda\left(A_j + \alpha_{j-\frac{1}{2}}I\right)U^+_{j-\frac{1}{2}}, \\ V_j^- &= \left(I + 2\lambda(\mathscr{A}_j - \alpha_{j-\frac{1}{2}}I)\right)U^-_{j-\frac{1}{2}} + 2\lambda\left(-\mathscr{A}_j + \alpha_{j+\frac{1}{2}}I\right)U^-_{j+\frac{1}{2}}, \\ V_j &= \left(I - \frac{1}{4}\Phi^+_{j+\frac{1}{2}} - \frac{1}{4}\Phi^-_{j-\frac{1}{2}}\right)U_j + \frac{1}{4}\Phi^+_{j+\frac{1}{2}}U_{j-1} + \frac{1}{4}\Phi^-_{j-\frac{1}{2}}U_{j+1}. \end{aligned} \tag{40}$$

Here we require that

$$\alpha_{j-\frac{1}{2}} \geqslant \max(\rho(A_j), \rho(\mathscr{A}_{j-1})). \tag{41}$$

The propagation speeds of waves involved in hyperbolic systems are finite i.e., those spectral radius are finite. Hence such α exists and can be calculated or determined by trial and error in numerical calculations. We also require the following CFL condition:

$$\lambda \max_U \rho\left(\frac{\partial F(U)}{\partial U}\right) \leqslant \frac{1}{4}. \tag{42}$$

Under (41), (42) $V^+ = \{V_j^+\}$ is a positive combination of U^+, $V^- = \{V_j^-\}$ is a positive combination of U^-, and $V = \{V_j\}$ is a positive combination of U of form (4) where conditions (i), (ii) and (iii) of (5) are satisfied. If (iv) of (5) were satisfied, we could conclude that

$$\|V^+\| \leqslant (1 + \mathrm{O}(\Delta))\|U^+\|, \quad \|V^-\| \leqslant (1 + \mathrm{O}(\Delta))\|U^-\|, \quad \|V\| \leqslant (1 + \mathrm{O}(\Delta))\|U\|. \tag{43}$$

By (39),

$$U_j^* = \frac{1}{4}V_j^+ + \frac{1}{4}V_j^- + \frac{1}{2}V_j.$$

Take the scalar product of it with U_j^*, then use the Schwartz inequality, we get

$$(U_j^*, U_j^*) \leqslant \frac{1}{4}(V_j^+, V_j^+) + \frac{1}{4}(V_j^-, V_j^-) + \frac{1}{2}(V_j, V_j).$$

Summing over j

$$\|U^*\|^2 \leqslant \frac{1}{4}\|V^+\|^2 + \frac{1}{4}\|V^-\|^2 + \frac{1}{2}\|V\|^2.$$

Using (43) we get

$$\|U^*\|^2 \leqslant (1+\mathrm{O}(\Delta))\left(\frac{1}{4}\|U^+\|^2+\frac{1}{4}\|U^-\|^2+\frac{1}{2}\|U\|^2\right).$$

Combining this with (35) we finally get that

$$\|U^*\| \leqslant (1+\mathrm{O}(\Delta))\|U\|, \tag{44}$$

under the conditions of (41), (42) and (iv) of (5).

2.3. Runge–Kutta time discretization

To achieve second order accuracy in time we use the following second order accurate Runge–Kutta method, due to Shu and Osher [16]:

$$\begin{aligned} U_j^* &= U_j^m - \frac{\Delta t}{\Delta x}\left[F^+_{j+\frac12} - F^+_{j-\frac12} + F^-_{j+\frac12} - F^-_{j-\frac12}\right],\\ U_j^{**} &= U_j^* - \frac{\Delta t}{\Delta x}\left[F^{+,*}_{j+\frac12} - F^{+,*}_{j-\frac12} + F^{-,*}_{j+\frac12} - F^{-,*}_{j-\frac12}\right],\\ U_j^{m+1} &= \frac{1}{2}U_j^m + \frac{1}{2}U_j^{**}. \end{aligned} \tag{45}$$

Here $F^{+,*}$ abbreviates the numerical flux F^+ evaluated at U^* and $F^{-,*}$ abbreviates the numerical flux F^- evaluated at U^*.

By the previous analysis,

$$\|U^{**}\|^2 \leqslant (1+\mathrm{O}(\Delta))\|U^*\|^2 \leqslant (1+\mathrm{O}(\Delta))\|U\|^2,$$

so

$$\|U^{m+1}\| \leqslant \frac{1}{2}\|U^m\| + \frac{1}{2}\|U^{**}\| \leqslant (1+\mathrm{O}(\Delta))\|U^m\|. \tag{46}$$

2.4. Positive schemes for multi-dimensional systems of conservation laws

For multi-dimensional systems, we use the simple and well-known dimension-by-dimension technique [16], which blends well with the positive schemes.

Consider multi-dimensional hyperbolic systems of conservation laws (1). Using the dimension-by-dimension technique [16], let positive schemes be of conservative form

$$U_J^* = U_J - \sum_{s=1}^{d} \frac{\Delta}{\Delta x_s}\left[F^+_{J+\frac12 e_s} - F^+_{J-\frac12 e_s} + F^-_{J+\frac12 e_s} - F^-_{J-\frac12 e_s}\right]. \tag{47}$$

A family of positive fluxes is constructed exactly as in one-dimension,

$$\begin{aligned} F^+_{J+\frac12 e_s} &= F_s^+\left(U^+_{J+\frac12 e_s}\right),\\ U^+_{J+\frac12 e_s} &= U_J + \frac{1}{2}\Phi^+_{J+\frac12 e_s}(U_J - U_{J-e_s}) = U_J + \frac{1}{2}\Psi^+_{J+\frac12 e_s}(U_{J+e_s} - U_J),\\ F^-_{J+\frac12 e_s} &= F_s^-\left(U^-_{J+\frac12 e_s}\right),\\ U^-_{J+\frac12 e_s} &= U_{J+e_s} - \frac{1}{2}\Phi^-_{J+\frac12 e_s}(U_{J+2e_s} - U_{J+e_s}) = U_{J+e_s} - \frac{1}{2}\Psi^-_{J+\frac12 e_s}(U_{J+e_s} - U_J). \end{aligned} \tag{48}$$

It is worth to note that the family of schemes (47), (48) uses component-wise limiting instead of field-by-filed limiting.

The family of schemes (47), (48) are positive under the following CFL condition:

$$\sum_{s=1}^{d} \frac{\Delta t}{\Delta x_s} \max_U \rho\left(\frac{\partial F_s(U)}{\partial U}\right) \leqslant \frac{1}{4}. \tag{49}$$

We use the same second order accurate energy preserving Runge–Kutta of Shu and Osher [16] to achieve second order accuracy in time: for $m = 0, 1, \ldots,$

$$\begin{aligned} U_J^* &= U_J^m - \sum_{s=1}^{d} \frac{\Delta t}{\Delta x_s}\left[F^+_{J+\frac{1}{2}e_s} - F^+_{J-\frac{1}{2}e_s} + F^-_{J+\frac{1}{2}e_s} - F^-_{J-\frac{1}{2}e_s}\right], \\ U_J^{**} &= U_J^* - \sum_{s=1}^{d} \frac{\Delta t}{\Delta x_s}\left[F^{+,*}_{J+\frac{1}{2}e_s} - F^{+,*}_{J-\frac{1}{2}e_s} + F^{-,*}_{J+\frac{1}{2}e_s} - F^{-,*}_{J-\frac{1}{2}e_s}\right], \\ U_J^{m+1} &= \frac{1}{2}U_J^m + \frac{1}{2}U_J^{**}. \end{aligned} \tag{50}$$

It is straight forward to extend our analysis (46) of l^2 boundedness to multi-dimensional hyperbolic systems:

$$\|U^{m+1}\| \leqslant \frac{1}{2}\|U^m\| + \frac{1}{2}\|U^{**}\| \leqslant (1 + \mathrm{O}(\Delta))\|U^m\|.$$

3. Numerical experiments

In all examples in the section, we use local flux splitting, see (17), with

$$\alpha_{j+\frac{1}{2}} = \mu \max\left(\rho\left(\frac{\partial F(U_j)}{\partial U}\right), \rho\left(\frac{\partial F(U_{j+1})}{\partial U}\right)\right). \tag{51}$$

Here μ is called the amplifier and $\mu = 1$ in example 1 and $\mu = 1.3$ in example 2. $\rho(A)$ denotes the spectral radius of the Jacobian matrix $A = (\partial F(U)/\partial U)$. Van Leer's limiter function is used.

We approximate solutions of the two-dimensional Euler equations of Gas Dynamics,

$$U_t + F_1(U)_x + F_2(U)_y = 0,$$

$$U = (\rho, m, n, E)^{\mathrm{T}},$$

$$F_1(U) = (m, \rho u^2 + P, \rho uv, u(E+P))^{\mathrm{T}},$$

$$F_2(U) = (n, \rho uv, \rho v^2 + P, v(E+P))^{\mathrm{T}},$$

$$P = (\gamma - 1)\left(E - \frac{1}{2}\rho(u^2 + v^2)\right),$$

$$m = \rho u, \quad n = \rho v.$$

Example 1: *Double Mach reflection.* A planar shock is incident on an oblique wedge at a 60° angle. The test problem involves a Mach 10 shock in air, $\gamma = 1.4$. The undisturbed air ahead of the shock has density 1.4 and pressure 1. We use the boundary conditions described in [23]. The flow at time $t = 0.2$, computed by the positive scheme of Kurganov and Tadmor, is plotted in Fig. 1 with $\Delta x = \Delta y = \frac{1}{120}$, $\frac{\Delta t}{\Delta x} = 0.02$. In each plot 30 equally spaced contours are shown.

There are three difficulties in computing this flow mentioned in [23]. The first difficulty is the rather weak second Mach shock; dies out entirely by the time it reaches the contact discontinuity from the first Mach reflection. Fig. 1 shows that the second Mach shock is perfectly captured. The second difficulty is the jet formed when the flow of the denser fluid is deflected by a pressure gradient built up in the region where the first contact discontinuity approaches the reflecting wall. Fig. 1 shows that the jet is extremely well captured. The third difficulty is caused by the region bounded by the second Mach shock, the curved reflected shock, and the reflecting wall. The double Mach reflection contains both steady and unsteady structures. The curved reflected shock is moving rapidly at its right end and is not moving at all at its left end; this causes oscillations for many difference schemes. Just as the original positive schemes, the positive scheme of Kurganov and Tadmor generates no oscillation at all; thus the positive scheme overcomes extremely well all three numerical difficulties.

Example 2: *A Mach 3 wind tunnel with a step.* This problem has been a useful test for schemes for many years. The tunnel is 3 length units long and 1 length unit wide, with a step which is 0.2 length units high and 0.6 length units away from the left end of the tunnel (see Fig. 2). The state behind the incoming shock has density 1.4, pressure 1.0, and velocity 3 from left to right. These are used as boundary condition at the left; at the right all horizontal gradients are assumed to vanish. Along the walls of the tunnel and the obstacle reflecting conditions are applied in the perpendicular direction. The corner of the step is the center of a rarefaction fan and hence is a singular point of the flow. At the corner we use the boundary condition used by Woodward and Colella in [23]. The amplifier $\mu = 1.3$ is used, see Eq. (51). This value determined by trial and errors. The density and pressure contours in the tunnel at time 4 are displayed in Fig. 2 with $\Delta x = \Delta y = \frac{1}{80}$, $\frac{\Delta t}{\Delta x} = 0.15$. The flow at time 4 is still unsteady.

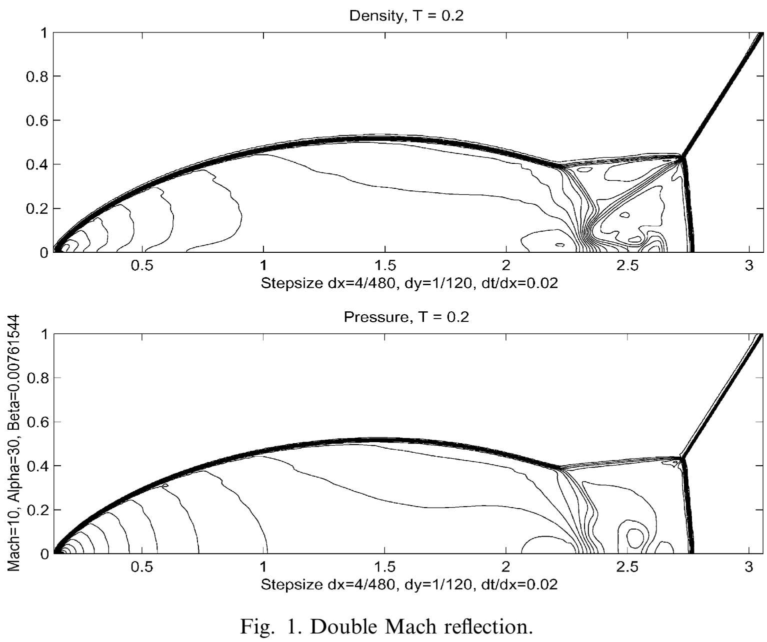

Fig. 1. Double Mach reflection.

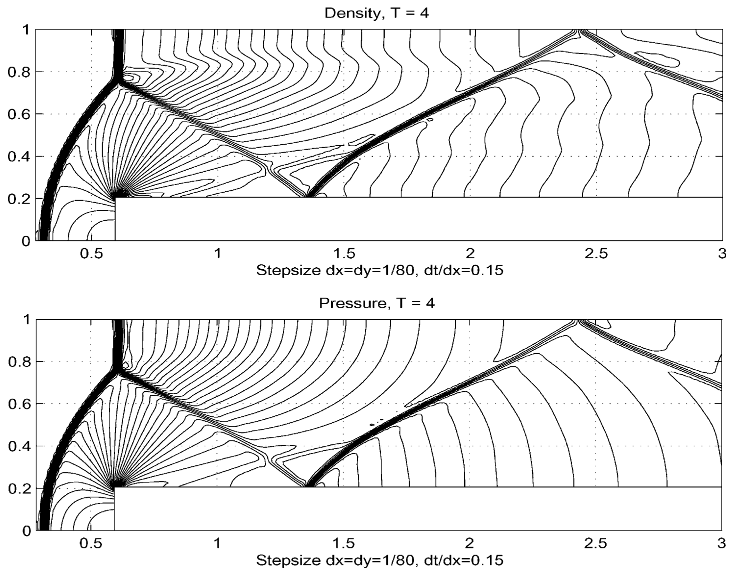

Fig. 2. Wind tunnel.

The general position and shape of the shocks are accurate. The shocks are well captured. There is no numerical noise, the contact discontinuities and the weak oblique shock are resolved. The resolution is as high as that of our original positive schemes [11].

4. Discussion

TVD (Total-Variation-Diminishing) is a proper principle for designing numerical schemes to solve hyperbolic equations (or linear hyperbolic systems) in one space dimension. For multi-dimensional hyperbolic systems, due to the possibility of focusing, TV norm is not bounded and hence no longer proper for such systems. The only functional known to be bounded for solutions of linear hyperbolic systems is L^2 norm [2]. Extending this result, [11], the positivity principle was introduced for multi-dimensional hyperbolic systems. The rationale of positivity principle is l^2 stability. Positivity principle is a proper designing principle for solving multi-dimensional hyperbolic systems.

Central schemes [6,7,9,10,13,14] are field-by-field limiting free. Other field-by-field limiting free schemes are [8] and [12]. Hence with helps of these schemes hyperbolic systems with complex eigensystems or weak hyperbolic systems can be solved easily and efficiently.

In this paper we show that Kurganov and Tadmor scheme, a field-by-field limiting free scheme, is positive for symmetric or symmetrizable multi-dimensional hyperbolic systems.

We present two numerical experiments to add to the ones carried by Kurganov and Tadmor [9]. The numerical resolutions we obtained are as high as the ones we got from our original positive schemes [11], which use field-by-field limiting.

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